Generic Uniqueness of the Solutions to a Continuous Linear Programming Problem

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February 14, 2005

Abstract

Let $f : R \rightarrow R$ be strictly increasing. We are interested in the set of probability distributions μ on the interval $[0, S]$ that solve the linear programming problem $\max_{\mu} \int_0^S f(p)\ d\mu(p)$ subject to $\int_0^S g(p)\ d\mu(p) \leq C$. We provide a sufficient condition on the pair (f, g) for the solution to the linear programming problem to be unique and show that this sufficient condition is satisfied "generically."

Keywords: Linear Programming

JEL number: C60

1 The problem

Let $[0, S]$ be an interval of the real line. Let the function $f : [0, S] \rightarrow [0, 1]$ be continuous and strictly increasing. Let the function $g : [0, S] \rightarrow \mathbf{R}$ be continuous. Let $\mathcal{M}$ denote the set of probability distributions on $[0, S]$. We wish to characterize the solutions to the following linear problem:

$$\max_{\mu \in \mathcal{M}} \int_0^S f(p)\, d\mu(p) \tag{P}$$
$$\text{s.t.} \int_0^S g(p)\, d\mu(p) \leq C.$$

To guarantee existence of a solution, in what follows we assume that $\min_{p \in [0,S]} g(p) \leq C$. We will prove the following result showing that, "generically," the solution to problem P is unique.

Proposition 1 *The solution to problem P is unique and puts positive probability on at most two points whenever $g \circ f^{-1}$ belongs to a set which is a dense $G - \delta$ set in the space of all continuous functions equipped with the supnorm.*

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2 Characterization of the solutions to problem P

Let $\mathcal{T}$ denote the set of continuous functions $x(t) : [0, S] \to \mathbf{R}$ with the property that at least one of the hyperplanes that supports $x(t)$ from below is tangent to $x(t)$ at more than two points. In what follows, the superscript ^{C} indicates set complements.

Lemma 2 *If $g \circ f^{-1}$ belongs to $\mathcal{T}^C$, any solution to problem P places all the probability on one or two points in* $[0, S]$.

Proof: If μ^* is a solution to problem P, there exists a number $\lambda \geq 0$ such that μ^* maximizes the Lagrangean

$$\mathcal{L}(\mu, \lambda) = \int_0^S [f(p) - \lambda g(p)] \, d\mu(p) + \lambda C.$$

The case $\lambda = 0$ is easily disposed of, since in that case the (unique) solution requires all the mass to be places on S. Let us turn, then, to the case $\lambda > 0$. If μ^* is to maximize the Lagrangean, the set of p's on which μ^* may place positive probability is given by

$$\mathcal{A} = \arg\max_p [f(p) - \lambda g(p)].$$

By definition of $\mathcal{A}$, there is a number M such that

$$\begin{aligned} f(p) - \lambda g(p) &= M \quad \text{for } p \in \mathcal{A} \\ f(p) - \lambda g(p) &< M \quad \text{for } p \notin \mathcal{A} \end{aligned}$$

If μ^* puts positive mass on more than two points, then the cardinality of $\mathcal{A}$ would have to exceed 2. Consider the transformation $\varphi(p) = f^{-1}(p/S)$. The function φ is a one-to-one mapping of $[0, S]$ onto itself. We can therefore write

$$\begin{aligned} f(\varphi(p)) - \lambda g(\varphi(p)) &= M \quad \text{for } \varphi(p) \in \mathcal{A} \\ f(\varphi(p)) - \lambda g(\varphi(p)) &< M \quad \text{for } \varphi(p) \notin \mathcal{A}, \end{aligned}$$

or, with the obvious meaning of symbols,

$$\begin{aligned} \tfrac{p}{S} - \lambda g(\varphi(p)) &= M \quad \text{for } p \in \varphi^{-1}(\mathcal{A}) \\ \tfrac{p}{S} - \lambda g(\varphi(p)) &< M \quad \text{for } p \notin \varphi^{-1}(\mathcal{A}). \end{aligned}$$

Note that the set $\varphi^{-1}(\mathcal{A})$ has the same cardinality of $\mathcal{A}$. Thus, if $\mathcal{A}$ has cardinality greater than 2, it means that the two numbers λ and M are such that the straight line identified by $\frac{1}{\lambda}\left(\frac{p}{S} - M\right)$ is tangent to the function $g\left(f^{-1}(p/S)\right)$ at more than two points and never exceeds it. This means that there is a tangent hyperplanes to the set

$$Y = \left\{(p, y) : y \leq g\left(f^{-1}(p/S)\right)\right\}$$

which makes contact with the set Y at more than two points. $\diamondsuit$

Lemma 3 *If $g \circ f^{-1}$ belongs to $\mathcal{T}^C$, the solution to problem P is unique.*

Proof: If the constraint is not binding then clearly μ^* is unique. If the constraint is binding, Lemma 2 guarantees that there are two points, call them p_L and p_H (here we allow for the possibility that $p_L = p_H$), such that

$$\mu^*(p_L) + \mu^*(p_H) = 1, \tag{1}$$

where the constraint in problem (P) reads

$$g(p_L) \cdot \mu^*(p_L) + g(p_H) \cdot \mu^*(p_H) = C. \tag{2}$$

The system of equations given by (1) and (2) identifies a unique pair $\mu^*(p_L), \mu^*(p_H)$ unless the system has less than full rank, i.e., unless $g(p_L) = g(p_H)$. But in that case, since f is strictly increasing, the (unique) solution requires all the mass to be placed on $\max\{p_L, p_H\}$. $\diamondsuit$

3 Characterization of the set $\mathcal{T}$

We now show that the set $\mathcal{T}$ is "small" in this topological sense: that its complement $\mathcal{T}^C$ is a dense $G - \delta$ set, i.e., it is dense and the countable intersection of sets that are open, in the space of continuous functions equipped with the supnorm.

To this end, let $C(x(\cdot))$ denote the convex hull of $x(\cdot)$, i.e., the function that is given by the intersection of all the epigraphs of the affine linear functions that lie below the function $x(t)$ and are tangent to it. Consider all open intervals ι of the real line such that for all $t \in \iota$, the convex hull $Cx(t)$ coincides with a linear affine function, and such that this property is not enjoyed by any open interval containing ι. Among those, select all those intervals ι' such that $x(t) - Cx(t) = 0$ for some $t \in \iota'$. If there are no such ι', then $x \notin \mathcal{T}$. If there are such intervals, let the extremes of the largest among these be denoted by a_x and b_x (if there are several largest intervals, pick one at random.) Clearly, $x(t)$ and $C(x(t))$ agree at $t = a_x, b_x$. Furthermore, denote

$$\begin{aligned} m_x &= \arg \min_{t \in (a_x, b_x)} \left| t - \frac{a_x + b_x}{2} \right| \\ &\quad \text{s.t. } x(t) - C(x(t)) = 0. \end{aligned}$$

Clearly, if $x \in \mathcal{T}$ we have $a_x < m_x < b_x$. The point m_x represents the point of tangency of the supporting hyperplane that is closest to the middle of the interval (a_x, b_x). Denote $I_x = [a_x, b_x]$ For $x \in \mathcal{T}$, I_x represents the largest interval at which any tangent hyperplane to x is tangent at more than two points.

Denote

$$\mathcal{T}_{n,m} = \left\{ x \in \mathcal{T} \text{ such that } (b_x - a_x) \geq \frac{1}{n} \text{ and } min \{m_x - a_x, b_x - m_x\} \geq \frac{1}{m} \right\}.$$

Clearly,

$$\mathcal{T} = \bigcup_{\substack{n \geq 1 \\ m \geq n}} \mathcal{T}_{n,m},$$

and so to obtain the required characterization for $\mathcal{T}^C$ we need to show that for each n, m, the set $(\mathcal{T}_{n,m})^C$ is open and that the set $\mathcal{T}^C$ is dense in the set of all continuous functions endowed with the supnorm. We start with showing dense.

Lemma 4 *The set $\mathcal{T}^C$ is dense in the set of continuous functions endowed with the supnorm.*

Proof: We show that in the supnorm neighborhood of every continuous function there is a function $x' \in \mathcal{T}^C$. Fix, then, an arbitrary continuous function x. If $x \in \mathcal{T}^C$ there is nothing to prove, so we can focus on the case $x \in \mathcal{T}$. Our task is to show that there is a function $\tilde{x}$ close to x with the property that no supporting hyperplane to $\tilde{x}$ has more than two contact points with $\tilde{x}$. Let $\mathcal{H}$ denote the set of hyperplanes that have more than two contact points with x. Elements of $\mathcal{H}$ are identified by their slope h. For every hyperplane $h \in \mathcal{H}$, take the sup and the inf of all its contact points t and call those a_h and b_h. Consider now a continuous function $d_g(t)$ which is equal to 0 for every t unless $t \in (a_h, b_h)$ for some $h \in \mathcal{H}$, in which case $d_g(t)$ assumes values strictly between zero and 1. Let $\tilde{x}_\varepsilon(t) \equiv [1 + \varepsilon \cdot d_g(t)] \cdot x(t)$. For any $\varepsilon > 0$, the function $\tilde{x}_\varepsilon(t)$ has exactly the same set of supporting hyperplanes as $x(t)$. Moreover, by construction no hyperplane is tangent to $\tilde{x}_\varepsilon(t)$ at more than two points. Since the function $\tilde{x}_\varepsilon(t)$ can be made arbitrarily close to $x(t)$ in the supnorm by choosing ε to be small, the set $\mathcal{T}^C$ is shown to be dense, as desired. $\diamondsuit$

Lemma 5 *For each $n \geq 1, m \geq n$, the set $(\mathcal{T}_{n,m})^C$ is open in the set of continuous functions endowed with the supnorm.*

Proof: The result will follow if we show that $\mathcal{T}_{n,m}$ is closed. To this end, let's fix an arbitrary sequence $\{x_i\}$ that converges to some x in the supnorm. For each x_i, consider the set $D_i =$

$\{a_{x_i}, m_{x_i}, b_{x_i}\}$. Since the interval $[0, S]$ is compact in the Euclidean distance, the space of subsets of $[0, S]$ endowed with the Hausdorff distance is compact (see Theorem 3.71 in Aliprantis and Border 1999). The sequence of sets $\{D_i\}$ therefore contains a convergent subsequence $\{D_j\}$. Consequently, the limits (in the euclidean distance) of a_{x_j}, m_{x_j}, and b_{x_j} for $j \to \infty$ exist. Denote these limit points by a, m, and b respectively. It is clear that $(b - a) \geq \frac{1}{n}$ and $min\{m - a, b - m\} \geq \frac{1}{m}$.

Since the mapping $C(\cdot)$ is continuous in the supnorm, we have

$$\lim_{j\to\infty} C(x_j(a)) = C(x(a)) \text{ and } \lim_{j\to\infty} C(x_j(b)) = C(x(b))$$

For each j, the function $C(x_j(t))$ is linear affine for $t \in \left[a_{x_j}, b_{x_j}\right]$. Thus, for $t \in [a, b]$, the function $C(x(t))$ is the limit in the supnorm of linear affine functions, hence a linear affine function itself. To finish the proof, we use the continuity of the mapping $C(\cdot)$ to write

$$x(m) = \lim_{j\to\infty} x_j\left(m_{x_j}\right) = \lim_{j\to\infty} C\left(x_j\left(m_{x_j}\right)\right) = C(x(m)).$$

We have shown that $C(x(t))$ is linear affine for $t \in [a, b]$, and that $x(m) = C(x(m))$. This shows that $x \in \mathcal{T}_{n,m}$, and thus that $\mathcal{T}_{n,m}$ is closed. $\diamondsuit$

4 Conclusion

We have shown that $\mathcal{T}^C = \bigcap\limits_{\substack{n\geq 1\\ m\geq n}} (\mathcal{T}_{n,m})^C$ is dense, and is the countable intersection of sets that are open, in the space of continuous functions endowed with the supnorm. This fact, in conjunction with with Lemma 3, proves Proposition 1.

References

[1] Aliprantis, Charalambos D., and Border, Kim C. *Infinite Dimensional Analysis.* 2nd ed. Springer Verlag, Heidelberg, 1999.