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Nonlinear Dynamics in Economics and Social Sciences

Proceedings of the Second Informal Workshop
Held at the Certosa di Pontignano, Siena, Italy
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Foreword

This volume constitutes the Proceedings of the "Nonlinear Dynamics in Economics and Social Sciences" Meeting held at the Certosa di Pontignano, Siena, on May 27-30, 1991.

The Meeting was organized by the National Group "Modelli Nonlineari in Economia e Dinamiche Complesse" of the Italian Ministery of University and Scientific Research, M.U.R.S.T.

The aim of the Conference, which followed a previous analogous initiative taking place in the very same Certosa, on January 1988*, was the one of offering a come together opportunity to economists interested in a new mathematical approach to the modelling of economical processes, through the use of more advanced analytical techniques, and mathematicians acting in the field of global dynamical systems theory and applications.

A basic underlying idea drove the organizers: the necessity of focusing on the use that recent methods and results, as those commonly referred to the overpopularized label of "Chaotic Dynamics", did find in the social sciences domain; and thus to check their actual relevance in the research program of modelling economic phenomena, in order to individuate and stress promising perspectives, as well as to curb excessive hopes and criticize not infrequent cases where research reduces to mechanical, ad hoc, applications of "à la mode" techniques.

In a word we felt the need of looking about the state of the arts in non-linear systems theory applications to economics and social processes: hence the title of the workshop and the volume.

The Meeting lasted four days. Mornings were devoted entirely to four minicourses given by R. Abraham, Economics and the Environment: Global Erodynamic Models; R.H. Day, Chaotic Dynamics in Micro and Macro Economic Processes; H.-W. Lorenz, Complextity in Deterministic, Nonlinear Business-Cycle Models; C. Mira, Toward a Knowledge of the Two-Dimensional Diffeormophism. Afternoons were taken by invited lectures and contributed papers.

About one hundred participants came from Europe and the Americas and more than 30 articles were presented.

Materials in the present volume are organized following the meeting structure.

The first section contains the notes of the minicourses given by R. Abraham, H.-W. Lorenz, within the text of one of the lectures given by C. Mira.

Lessons by R.H. Day related essentially to his joint paper with G.Pianigiani "Statistical Dynamics and Economics", (J. of Economic Behaviour and Organization, volume 16, July 1991, pp.37-84), to which we refer the interested reader.

In the second part the texts of three invited lectures, by W. Böhm, R. Goodwin and A.G. Malliaris, are published. Even if these articles were prepared on the occasion of the meeting, only the first has been formally presented there.

Finally, the third section is devoted to thirteen contributed papers presented in Pontignano, which the authors submitted for publication and were positively refereed.

It is our feeling that the meeting was really successfull in attaining its intended goals and we would like to express our gratitude to the invited speakers for the quality of the lectures they delivered in Pontignano, and to all the participants for their highly interesting contributions, the lively discussions and the stimulating and friendly atmosphere they were able to create.

We would also like to express our thanks to the many referees for their important help in selecting the papers to be included in this volume.

The University of Florence and the University of Siena jointly sponsored the meeting and we gratefully acknowledge their scientific and financial support, as well as the contributions which the Organizing Committee received from the Monte dei Paschi di Siena Bank.

A final particular thank goes to Mrss. Marcella Dragoni and Bianca Maria Fabrini, whose secretary work during the Meeting and the editing process has been invaluable.

Franco Gori
Lucio Geronazzo
Marcello Galeotti

* M. Galeotti, L. Geronazzo, F. Gori, *Non Linear Dynamics in Economics and Social Sciences*, Pitagora Editrice, Bologna, 1990.

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Acyclicity of Optimal Paths

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Abstract

It is well known that the stability of optimal paths for infinite horizon concave problems is not assured for all discount factors. Along the ideas developed in Boldrin and Montrucchio (IER, 1988), we provide new results of stability which are related to the notion of acyclicity. Some order relations are introduced which can be seen as generalizations of Liapunov theory. Our results are quite complete for one-dimensional case. In higher dimensions we state some new results.

1 Introduction

This paper is concerned with the dynamics of optimal policy functions. The matter is interesting since the contributions of some authors (see for example Boldrin–Montrucchio [2], where it is shown that any kind of erratic behaviour is possible, provided the discount factor is low enough). Classical turnpike theorems, on the other hand, show that regular behaviour is forced when the discount factor gets close to one. In the middle there should be some kind of bifurcation, but little is known in general.

The purpose of this paper is to give results about the regular behaviour of the policy function, based on properties of the primitives of the problem, and independently of the magnitude of the discount factor.

The results obtained are based on the notion of acyclicity of a binary relation, and lie in the line of research started by Montrucchio [7] and developed in Boldrin and Montrucchio [3]. One of the advantages of this approach is that the analysis is carried out without assumptions of differentiability, although further characterizations can be provided in case the primitives are differentiable.

The results we obtain are quite complete for the one-dimensional case and, when compared with the alternative unimodularity approach, prove more general.

In Section 2 we recall some brief definitions concerning dynamical systems. Section 3 is devoted to the exposition of a general theorem and of additional results for one-dimensional systems. Section 4 contains the application of the theorems above to the standard problem of dynamic programming over infinite horizon. The results about the dynamics of the policy function are derived, and the one-dimensional case is examined in detail. Furthermore, we show the connections between acyclicity and unimodularity. In the last subsection we show how, using our method, it is possible to allow only low-period cycles in one dimension, ruling out more complicated dynamics.

* The first author was partially supported by a grant from M.U.R.S.T., National Group on Nonlinear Dynamics in Economics and Social Sciences

2 Definitions and notation

In this section we will develope some notation, and revise some knowledge that we will n later on.

Suppose X is a complete metric space. A discrete dynamical system is a continuous mappi $f : X \to X$. It is well known that the solution to the initial value problem $x_t = f(x_{t-1})$, $x_0 = x^0$ is given by $x_t = f^t(x^0)$ where f^t denotes the t-th iterate of f.

Denote with

$$\mathrm{Fix}(f) = \{x \in X; f(x) = x\}$$

and

$$\mathrm{Per}(f) = \{x \in X; f^n(x) = x \text{ for some } n \in \mathrm{N}\}.$$

Let us recall some classical definitions related to the asymptotic behaviour of the trajectories $f^n(x)$.

For any $x \in X$ define the ω-limit set $\omega(x)$ as the set of all limit points of the sequence $f^n(x)$. An alternative definition (see [6]) is:

$$\omega(x) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^k(x)}.$$

The basic properties of limit sets are summarized in

Fact 1 *Every limit set $\omega(x)$ is closed and positively invariant. If in addition the sequence $f^n(x)$ is relatively compact, then $\omega(x)$ is nonempty, compact, invariant and invariantly connected.*

We refer to [5, 6] for details. Here, we just recall that a set $H \subset X$ is said to be *positively invariant* if $f(H) \subseteq H$, and *invariant* if $f(H) = H$.

Furthermore, H is said to be *invariantly connected* if it is not the union of two nonempty, disjoint closed invariant sets. It should be noted that the motion $f^n(x)$ is relatively compact when X is compact. Analogoulsy it is possible to introduce the α-limit set. Given $x \in X$, the set $\alpha(x)$ is defined as the collection of points $y \in X$ such that there exist two sequences x_n and $\alpha_n \geq 0$ satysfying

$$x_0 = x, \quad x_n \to y \quad \text{as } n \to +\infty \quad \text{and } f^{\alpha_n}(x_{n+1}) = x_n.$$

In the sequel we shall denote

$$\omega(f) = \cup_{x \in X} \omega(x) \text{ and } \alpha(f) = \cup_{x \in X} \alpha(x).$$

Let us introduce the second concept which is related to the recurrence property.

Defini on 1 *A point $p \in X$ is a* **wandering point** *for f if there exists a neighbourhood V of p and sitive integer t_0 such that $f^t(V) \cap V = \emptyset$, for all $t > t_0$. Otherwise we say that p is a non* **dering point**. *The collection of all such p's is the* **non–wandering set** $\Omega(f)$.

Since e X is metric, we can give a simpler definition for $\Omega(f)$. A point x is in $\Omega(f)$ iff th two sequences $x_k \to x$ and $y_k \to x$ as $k \to +\infty$, and a sequence of integers n_k — that $f^{n_k}(x_k) = y_k$.

F $\in X$ it is possible to write the following relation:

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}(f) \subseteq \omega(f) \cup \alpha(f) \subseteq \Omega(f).$$

A classical discussion of the set Ω for discrete and continuous dynamical systems is in [9]. Informally, we can say that Ω contains equilibria, α-limits and periodic orbits, and also all other points of X which keep coming back near themselves under iteration of f: Ω is the dynamically interesting part of X. If we were able to show that $\Omega(f) = \text{Fix}(f)$, then we would say that the map f is "simple", in the sense that no periodic points, "chaos" or topologic transitivity is allowed. If $\Omega(f) = \text{Fix}(f)$ and $\text{Fix}(f)$ is finite then $\omega(x) = \{p\}$ for some $p \in \text{Fix}(f)$. This follows from Fact 1 in addition with the remark that a finite invariantly connected set is a periodic orbit.

Now, let us turn to some algebraic notion we use in the sequel. We define a binary relation over X, called P; that is, P is a subset of $X \times X$. We will write yPx for $(x, y) \in P$. Moreover, thinking of P as a correspondence, we will denote with $P(x)$ the set $P(x) = \{y \in X : yPx\}$, and we will feel free to use the terminology of correspondences. For example, we will say that P is lower-hemicontinuous if the correspondence defined by P is lower-hemicontinuous.

Now, let us turn to the notion of *acyclicity.*

Definition 2 *A binary relation P over X is called* cyclic *if for some $n \geq 2$ there exists a sequence $(x_1 \ldots x_n)$ of distinct points in X such that x_2Px_1, $x_3Px_2, \ldots$, x_nPx_{n-1}, x_1Px_n. P is said to be* acyclic *if it is not cyclic.*

REMARK: If we define the *transitive completion* as the set $P^* = \{(x, y) \in X \times X$ such that there exists a finite sequence $(x_1 \ldots x_n)$ of points in X satisfying x_1Px, $x_2Px_1, \ldots$, x_nPx_{n-1}, yPx_n $\}$, then Definition 2 simply states that P is acyclic if and only if P^* is irreflexive. ◇

3 Binary relations

3.1 The general case

We give immediately the main theorem of this paper.

Theorem 3.1 *Let $f : X \to X$ be a continuous map and P a binary relation over X. Assume that f and P satisfy the following conditions*

1. *$f(x)Px$ for all $x \in X$ such that $f(x) \neq x$*

2. *P is open and acyclic.*

Then we have:

$$\Omega(f) = Fix(f).$$

Proof. See [3] and [7]: □

Theorem 3.1 might well be interpreted as a fixed-point theorem whenever X is compact. In fact, by Lemma 1 we are sure $\omega(f)$ is nonempty, and since the theorem establishes the equality $\omega(f) = \text{Fix}(f)$, we know that in this case the set of fixed points is nonempty, without requiring any convexity of the space X.

3.2 The one–dimensional case

In this subsection we are concerned with the case when X is an interval of the real line. The one–dimensional case yelds particularly good results with our approach, due to the completeness of the natural order structure on $\mathbf{R}$, and to the use of Sarkovsky's powerful theorem (see [8]) on the ranking of cycles.

Let us assume the following regularity conditions for the relation P.

i) P is lower-hemicontinuous.

ii) The set $P(x)$ is convex for every $x \in X$.

iii) If, for a particular point $x' \in X, P(x') = \emptyset$, then for any sequence $x_n \to x'$ such that $P(x_n) \neq \emptyset$ there exists a sequence $y_n \to x'$ with $y_n \in P(x_n)$.

Definition 3 *A binary relation P on X is said to be* n–cyclic *if there exists a set of distinct points $x_1 \dots x_n \in X$ such that $x_2Px_1, x_3Px_2, \dots x_nPx_{n-1}$, x_1Px_n. The relation will be called* n-acyclic *if it is not n-cyclic.*

Lemma 1 *Assume a binary relation P satisfies the regularity conditions i),ii),iii). Then for every sequence of distinct points $\{x_1 \dots x_n\} \in X$, such that $x_{i+1}Px_i$ $(n+1=1)$ there exists a continuous map $\mu : X \to X$ with the following properties:*

1. $\mu(x_i) = x_{i+1}$
2. $\mu(x)Px$ *for every* $x \neq \mu(x)$

Proof: See [3]. □

Theorem 3.2 *Suppose P is m–acyclic, and P satisfies Assumptions i)-iii) above. Then P is n–acyclic for $n \succ m$ in the Sarkovsky ordering of the natural numbers.*

Proof: Assume P is m–acyclic, and suppose there exists a cycle $y_1, \dots y_n$ with $y_{i+1}Py_i\,(n+1 = 1)$ and with $n \succ m$.

If P satisfies i)–iii) then, by Lemma 1, a continuous map $\mu : X \to X$ exists such that $\mu(y_i) = y_{i+1}$ and $\mu(x)Px$. But then we can apply Sarkovsky's theorem. In fact, the existence of a period–n cycle forces the existence of a period–m cycle, denoted by $\{x_1 \dots x_m\}$. But this, by Lemma 1, implies x_2Px_1, $x_3Px_2, \dots x_mPx_{m-1}$ and x_1Px_m, contradicting the assumption. □

The consequence of Theorem 3.2 is that it is sufficient to exclude 2-acyclicity in order to rule out acyclicity.

4 Optimal paths

4.1 The problem

The optimization problem is the usual one, described by (1) and by assumptions A1—A3 below (see for example [10]).

$$W_\delta(x) = \max \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t$$

$$\text{s. t. } (x_t, x_{t+1}) \in D \tag{1}$$
$$x_0 \text{ given in } X$$

A1 $X \subset \mathbf{R}^n$ is compact and convex.

A2 D is a convex subset of $X \times X$, and the correspondence $D : X \to X$ defined by $D(x) = \{y \in X : (x, y) \in D\}$ is a continuous and compact-valued correspondence, such that $x \in D(x)$ for all $x \in X$.

A3 $V : X \times X \to \mathbf{R}$ is a continuous and concave function, and $V(x, \cdot)$ is strictly concave for every $x \in X$.

We recall here some well-known properties of (1) that we will use in the sequel.

The function $W_\delta : X \to \mathbf{R}$ is the value function associated to (1). It is concave and continuous, and satisfies the Bellman equation

$$W_\delta(x) = \max_{y \in D(x)} \{V(x, y) + \delta W_\delta(y)\}. \tag{2}$$

Define $\tau_\delta : X \to X$ the continuous mapping solving (2), i. e.

$$W_\delta(x) = V(x, \tau_\delta(x)) + \delta W_\delta(\tau_\delta(x)). \tag{3}$$

We call τ_δ the *policy function* of (1), and we will often omit in what follows the subscript δ. It can be shown, using Bellman's optimality principle, that $\{x_t\}_{t=0}^{\infty}$ is a feasible sequence realizing the maximum of (1) if and only if it satisfies $x_{t+1} = \tau_\delta(x_t)$. The hypotheses made above have the goal of assuring the existence of a well-behaved policy function (for details see [10]).

It is well known (see [2]) that very complicated dynamic behaviour is possible for the function τ. Then it is interesting to obtain stability conditions for τ that do not depend on the explicit knowledge of τ.

4.2 Bellman's equation: the n-dimensional case

We will now provide the application of the general theorems provided in Section 3. Informally, we start from the fact that the function $\tau(x)$ is seen to be the maximizer of the right-hand side of (2), and is therefore "preferred" to any other point in X, including the linear combination of itself with the point x.

Definition 4 *Let $U : X \times X \to \mathbf{R}$ be continuous, with U, X and D satisfying A1-A3. For a given number $\theta \in [0, 1)$ define the binary relation P_θ over X as*

$$y P_\theta x \Leftrightarrow U(x, (1-\theta)x + \theta y) < U(x, y)$$

for $(x, y) \in X \times X$.

The relation P_θ is open in $X \times X$ because U is continuous.

We are interested in the case in which the function $U(x, y)$ is defined as

$$U(x, y) = V(x, y) + \delta W_\delta(y),$$

where V and W_δ are as in problem (1). In other words, one wants to study the acyclicity of the relation

$$yP_\theta x \Leftrightarrow V(x,(1-\theta)x+\theta y)+\delta W_\delta((1-\theta)x+\theta y) < V(x,y)+\delta W_\delta(y) \tag{4}$$

for $\theta \in [0,1)$, and $(x,y) \in X \times X$.

Since $\tau(x) = \text{argmax}_{y \in D(x)} U(x,y)$ for some suitable U, it is true that $\tau(x)P_\theta x$ when $\tau(x) \neq x$, because of the strict concavity of $U(x,\cdot)$, and because $(x,x) \in D$. We are then in the conditions to apply Theorem 3.1, except for the acyclicity of the relation P_θ.

Before turning to the acyclicity of P_θ, let us establish some properties of this relation following [4], where it was first introduced.

Proposition 1 *$\theta > \theta'$ implies $P_\theta \subset P_{\theta'}$, where P_θ is now considered as a subset of $X \times X$.*

Proof: Let $(x,y) \in P_\theta$. Then $U(x,(1-\theta)x+\theta y) < U(x,y)$. For fixed x and y, consider the function $\phi : [0,1) \to \mathbf{R}$ defined as $\phi(\theta) = U(x,(1-\theta)x+\theta y)$. Then $(x,y) \in P_\theta$ is equivalent to $\phi(\theta) < \phi(1)$. The concavity of $U(x,\cdot)$ guarantees that ϕ is concave, and therefore $\theta > \theta'$ implies $\phi(\theta') \leq \phi(\theta) < 1$, i. e. , $U(x,(1-\theta')x+\theta' y) < U(x,y)$, and thus $(x,y) \in P_{\theta'}$. □

Corollary 4.1 *If P_θ, defined as in Definition 4, is acyclic for $\theta = \bar{\theta}$, then it is also acyclic for all $1 > \theta > \bar{\theta}$.*

In the following we will convene to call θ-acyclic any U that induces a relation P_θ which is acyclic.

Looking at Definition 4 it is immediate to verify that

Proposition 2 *If V is θ-acyclic for some $\theta \in [0,1)$ then the new function $U(x,y) = V(x,y) + \psi(x)$ is also θ-acyclic for any function $\psi : X \to \mathbf{R}$.*

Definition 5 *Let $V(x,y)$ be as in A3. We say that it is* additively θ-acyclic *iff the functions $U(x,y) = V(x,y) + W(y)$ are θ-acyclic for any concave $W(\cdot)$.*

We know that the policy function satisfies (4) for any admissible θ, whenever $\tau(x) \neq x$, because of assumptions A2 on D. Then Theorem 3.1 applies. Obviously, if V is additively θ-acyclic for some $\theta \in [0,1)$, then the policy function τ is "simple" *for every $\delta \in [0,1)$*. Let us now turn to a sufficient condition that guarantees the additively θ-acyclicity of V.

Theorem 4.2 *Let $V(x,y)$ be as in A3, and τ_δ be as defined in (3). If there exists a θ such that for any N and for any sequence of distinct points $\{x_1 \ldots x_N\}$ one has*

$$\sum_{t=1}^{N} V(x_t,(1-\theta)x_t+\theta x_{t+1}) \geq \sum_{t=1}^{N} V(x_t,x_{t+1}) \tag{5}$$

with $x_{N+1} = x_1$, then V is additively θ-acyclic and τ_δ is "simple".

Proof: We will proceed by contradiction. Let (5) be satisfied for some $0 \leq \theta < 1$ and assume that P_θ as defined in (4) is not acyclic for some concave W. This means that there exists a sequence of points $\{x_1 \ldots x_N\}$ such that $x_{t+1}P_\theta x_t$ with $t = 1 \ldots N$ and $x_{N+1} = x_1$. In other words, for the given θ and chosen W one has

$$V(x_t,(1-\theta)x_t+\theta x_{t+1}) + W((1-\theta)x_t+\theta x_{t+1}) < V(x_t,x_{t+1}) + W(x_{t+1}) \qquad \text{for t=1...N.}$$

Summing over t:

$$\sum_{t=1}^{N}[V(x_t,(1-\theta)x_t+\theta x_{t+1})+W((1-\theta)x_t+\theta x_{t+1})]<\sum_{t=1}^{N}[V(x_t,x_{t+1})+W(x_{t+1})]. \qquad (6)$$

As W was assumed concave, we have

$$\sum_{t=1}^{N}W((1-\theta)x_t+\theta x_{t+1})\geq(1-\theta)\sum_{t=1}^{N}W(x_t)+\theta\sum_{t=1}^{N}W(x_{t+1})=\sum_{t=1}^{N}W(x_t).$$

The latter inequality implies that Equation (6) may be rewritten as:

$$\sum_{t=1}^{N}V(x_t,(1-\theta)x_t+\theta x_{t+1})<\sum_{t=1}^{N}V(x_t,x_{t+1})$$

which contradicts the hypothesis (5). □

Remark: Theorem 2 of [3] is obtained as a special case of Theorem 4.2 by taking $\theta=0$. ◇

Obviously, we have the following propositions.

Proposition 3 *If a function V satisfies condition (5) for $\theta=\bar{\theta}$, then V satisfies condition (5) for all $\theta>\bar{\theta}$.*

Proof: The same argument used in the proof of Proposition 1 may be applied to the function $\phi:[0,1)\to\mathbf{R}$ defined by $\phi(\theta)=\sum_{t=1}^{N}[V(x_t,(1-\theta)x_t+\theta x_{t+1})]$ for every given sequence $\{x_1\dots x_N\}$. □

Proposition 4 *If a function V satisfies (5) for some $\theta\in[0,1)$ then for any function $\psi:X\to\mathbf{R}$ and any concave function $\phi:X\to\mathbf{R}$, the new function $V^+(x,y)=V(x,y)+\psi(x)+\phi(y)$ also satisfies condition (5).*

Proof: One needs only to replicate the argument in the proof of Theorem 4.2. □

Proposition 4 may be quite misleading: in fact, the addition of a function $\phi(y)$ may render additively θ-acyclic a function that is not θ-acyclic. To show this, we have to remind the notion of α-concavity.

A function $V(x,\cdot)$ is said to be α-concave if

$$V(x,(1-\theta)y+\theta z)\geq(1-\theta)V(x,y)+\theta V(x,z)+\frac{1}{2}\alpha\theta(1-\theta)\|y-z\|^2,$$

for all $\theta in[0,1]$, y,z.

Proposition 5 *Take a $V(x,y)$ such that $V(x,\cdot)$ is α-concave for some $\alpha>0$. Suppose*

$$\sum_{t=1}^{N}V(x_t,x_t)\geq\sum_{t=1}^{N}V(x_t,x_{t+1})-\frac{1}{2}\alpha_1\sum_{t=1}^{N}\|x_t-x_{t+1}\|^2 \qquad (7)$$

for some $0<\alpha_1<\alpha$, for every N and for every sequence of distinct points $\{x_1\dots x_N\}$ in X with $x_{N+1}=x_1$. Then $V(x,y)$ turns out to be additively θ-acyclic for $\theta=\alpha_1/\alpha$.

Proof. By contradiction as usual, let $\{x_1 \dots x_N\}$ be a sequence of distinct points in X with $x_{N+1} = x_1$ which realizes a cycle for $\theta = \alpha_1/\alpha$. Then we have

$$V(x_t, (1-\theta)x_t + \theta x_{t+1}) + W((1-\theta)x_t + \theta x_{t+1}) < V(x_t, x_{t+1}) + W(x_{t+1}) \quad \text{for i=1...N.} \tag{8}$$

If $V(x, \cdot)$ is α-concave,

$$V(x_t, (1-\theta)x_t + \theta x_{t+1}) \geq (1-\theta)V(x_t, x_t) + \theta V(x_t, x_{t+1}) + \frac{1}{2}\alpha\theta(1-\theta)\|x_t - x_{t+1}\|^2,$$

and comparing with (8)

$$(1-\theta)V(x_t, x_t) + \theta V(x_t, x_{t+1}) + \frac{1}{2}\alpha\theta(1-\theta)\|x_t - x_{t+1}\|^2 + W((1-\theta)x_t + \theta x_{t+1}) < V(x_t, x_{t+1}) + W(x_{t+1}),$$

or

$$V(x_t, x_t) + W((1-\theta)x_t + \theta x_{t+1}) < V(x_t, x_{t+1}) - \frac{1}{2}\alpha\theta\|x_t - x_{t+1}\|^2 + W(x_{t+1}).$$

Summing up over t, because of the concavity of W, we have

$$\sum_{t=1}^{N} V(x_t, x_t) < \sum_{t=1}^{N} V(x_t, x_{t+1}) - \frac{1}{2}\alpha\theta \sum_{t=1}^{N} \|x_t - x_{t+1}\|^2$$

which contradicts (7), being $\alpha\theta = \alpha(\alpha_1/\alpha) = \alpha_1$. □

Finally, if V is differentiable in the second set of arguments, we have the following criterion to detect acyclicity, which is very useful from the computational point of view (we use the notation $V_2(x, y) \equiv \nabla_y V(x, y)$).

Proposition 6 *Assume V is as in A3, and differentiable in the second set of arguments. Then V is additively θ-acyclic for some $0 < \theta < 1$ if and only if*

$$\sum_{t=1}^{N} V_2(x_t, x_{t+1}) \cdot (x_{t+1} - x_t) \leq 0 \tag{9}$$

for every sequence of distinct points $\{x_1 \dots x_N\}$ in X with $x_{N+1} = x_1$, for every N.

Proof. Let the set $\{x_1 \dots x_N\}$ satisfying the assumptions of the proposition be given in X, and set $\phi(\theta) = \sum_{t=1}^{N} V(x_t, \theta x_{t+1} + (1-\theta)x_t)$. ϕ is strictly concave on $[0, 1]$ since $V(x, \cdot)$ is concave by assumption. Also, $\phi(\theta) \geq \phi(1)$ because V is acyclic. Hence $\phi\prime(1) \leq 0$, which is

$$\phi\prime(1) = \sum_{t=1}^{N} V_2(x_t, x_{t+1}) \cdot (x_{t+1} - x_t) \leq 0.$$

Conversely, suppose that (9) is satisfied for the generic suitable $\{x_1 \dots x_N\}$. Since $V(x, \cdot)$ is concave and differentiable, we have

$$V(x_t, x_{t+1}) \leq V(x_t, \theta x_{t+1} + (1-\theta)x_t) + (1-\theta)V_2(x_t, \theta x_{t+1} + (1-\theta)x_t)(x_{t+1} - x_t).$$

Summing up over t we have

$$\sum_{t=1}^{N} V(x_t, x_{t+1}) \leq \sum_{t=1}^{N} V(x_t, \theta x_{t+1} + (1-\theta)x_t) + Q$$

where

$$Q = (1-\theta)\left[\sum_{t=1}^{N} V_2(x_t, \theta x_{t+1} + (1-\theta)x_t)(x_{t+1} - x_t)\right].$$

But the quantity between brackets is nonpositive by (9), and so

$$\sum_{t=1}^{N} V(x_t, x_{t+1}) \leq \sum_{t=1}^{N} V(x_t, \theta x_{t+1} + (1-\theta)x_t)$$

holds, which implies the additively θ-acyclicity of V. □

Example 1. Take X a convex subset of $\mathbf{R}^n$, and $D = X \times X$. Let $V(x,y) = \phi(x) + \psi(y) + \mu\langle x, y\rangle$, with $\phi(x)$ α-concave on X, $\psi(y)$ β-concave on X and $\langle\cdot,\cdot\rangle$ denotes the inner product.

One may easily check that V will be concave iff $\mu^2 \leq \alpha\beta$, and will be strictly concave in the second argument if $\beta > 0$.

If $\mu \geq 0$, V is 0-acyclic. In fact, exploiting the relation

$$\|x_t - x_{t+1}\|^2 = \|x_t\|^2 + \|x_{t+1}\|^2 - 2\langle x_t, x_{t+1}\rangle$$

we get

$$\mu \sum_{t=1}^{N} \langle x_t, x_{t+1}\rangle \leq \mu \sum_{t=1}^{N} \|x_t\|^2,$$

which is the sufficient condition (5) if $\mu \geq 0$ (with $\theta = 0$).

If $\mu < 0$ one has, always using the above relation,

$$\mu \sum_{t=1}^{N} \langle x_t, x_{t+1}\rangle = \mu \sum_{t=1}^{N} \|x_t\|^2 - \frac{\mu}{2} \sum_{t=1}^{N} \|x_t - x_{t+1}\|^2,$$

or

$$\mu \sum_{t=1}^{N} \|x_t\|^2 = \mu \sum_{t=1}^{N} \langle x_t, x_{t+1}\rangle + \frac{\mu}{2} \sum_{t=1}^{N} \|x_t - x_{t+1}\|^2.$$

Then, if $-\beta < \mu$, V is acyclic with $\theta = |\mu|/\beta$ by Proposition 5. ◇

4.3 Bellman's equation: the one–dimensional case

As in the n–dimensional case, the relation we will consider is

$$yP_\theta x \Leftrightarrow V(x, (1-\theta)x + \theta y) + \delta W_\delta((1-\theta)x + \theta y) < V(x,y) + \delta W_\delta(y). \qquad (10)$$

But now, due to Theorem 3.2, we have more information. For example, we need not check a total acyclicity of the relation P_θ, since it is enough to rule out odd periodic points to ensure a certain degree of dynamical regularity of the map τ. The icing on the cake is that, to rule out periodic points of any period, it is sufficient to check that the relation P_θ is not 2-periodic. Making use of Theorem 4.2 we get

Proposition 7 *Let X be an interval in* **R**. *If, for some $\theta \in [0,1)$, V satisfies*

$$V(x,(1-\theta)x+\theta y)+V(y,(1-\theta)y+\theta x)\geq V(x,y)+V(y,x)$$

for all $(x,y),(y,x)\in X\times X$ with $x\neq y$, then V is additively θ-acyclic.

Proof: We are going to use Theorem (3.2). First, one must check that P satisfies the regularity conditions i)-iii) of Subsection 3.2.

i) is satisfied because of the continuity of V (and consequently of W).

ii) is true because of the concavity of $V(x,\cdot)$.

iii) holds because the strict concavity of V and the fact that $\theta<1$ imply that $P(x)\neq 0$ for all $x\in X$.

Now the proof is immediate, stemming from the comparison of Theorem 4.2 and Theorem 3.2. □

Now, let us define the *unimodularity property*, and show its connections with acyclicity (see [4]).

Definition 6 *A function $V(x,y)$ defined over $X\times X$ is called*

- **supermodular** *if for any two pairs x_1,x_2 and y_1,y_2 in X with $x_1\leq x_2$, $y_1\leq y_2$ we have* $V(x_1,y_1)+V(x_2,y_2)\geq V(x_1,y_2)+V(x_2,y_1)$
- **submodular** *if in the same situation we have* $V(x_1,y_1)+V(x_2,y_2)\leq V(x_1,y_2)+V(x_2,y_1)$

Proposition 8 *Let X be an interval, and $V:X\times X\to$* **R** *be unimodular. Then*

- *if V is supermodular the policy function τ_δ is non-decreasing*
- *if V is submodular the policy function τ_δ is non-increasing*

Proof: One should note, first of all, that $V(x,y)$ unimodular implies that $U(x,y)=V(x,y)+\delta W_\delta(y)$ is also unimodular. Let us consider the supermodular case, the other one being completely symmetric. Set $x_1<x_2$ and suppose, by contradiction, that $\tau_\delta(x_1)>\tau_\delta(x_2)$. Set $y_2=\tau_\delta(x_1)>y_1=\tau_\delta(x_2)$. Supermodularity gives: $U(x_1,\tau_\delta(x_2))+U(x_2,\tau_\delta(x_1))\geq U(x_1,\tau_\delta(x_1))+U(x_2,\tau_\delta(x_2))$. Strict concavity together with the optimality principle give

$$U(x_1,\tau_\delta(x_1))+U(x_2,\tau_\delta(x_2))>U(x_1,\tau_\delta(x_2))+U(x_2,\tau_\delta(x_1)),$$

a contradiction. □

We are now ready to show that supermodularity implies acyclicity, that is, acyclicity is a weaker condition that permits to have regular behaviours.

Proposition 9 *Consider problem 1 with $dim(X)=1$. In this case, if V is supermodular then it is additively 0-acyclic.*

Proof: We need only show that the conditions of Proposition 7 are satisfied. Let x and y be two points in X with, say, $x<y$. In the definition of supermodularity set $x_1=y_1=x$ and $x_2=y_2=y$.

This gives $V(x,x)+V(y,y)\geq V(x,y)+V(y,x)$, which is the desired inequality. □

REMARK: The converse is not true. See Example 2 ◇

An important class of return functions is that of the functions $V(x,y) = U(f(x) - g(y))$, where U is concave and f and g are nondecreasing functions. The relevance of this class of one–period return functions is that it covers the standard model of one–sector growth. It turns out that functions in this class are supermodular.

To prove our claim, suppose $x_1 \leq x_2$. One has

$$f(x_2) - g(y_1) \geq f(x_1) - g(y_1) \geq f(x_1) - g(y_2)$$

and

$$f(x_2) - g(y_1) \geq f(x_2) - g(y_2) \geq f(x_1) - g(y_2).$$

If we denote $\Delta f = f(x_2) - f(x_1)$, using the above inequalities and the fact that U is concave we get

$$\frac{U(f(x_2) - g(y_1)) - U(f(x_1) - g(y_1))}{\Delta f} \leq \frac{U(f(x_2) - g(y_2)) - U(f(x_1) - g(y_2))}{\Delta f},$$

or

$$U(f(x_2) - g(y_1)) + U(f(x_1) - g(y_2)) \leq U(f(x_2) - g(y_2)) + U(f(x_1) - g(y_1)),$$

which is the definition of supermodularity.

In the case of differentiability, we can say (see Proposition 6)

Proposition 10 *Assume V is as in A3 and C^1. Then V is additively θ-acyclic for some $0 < \theta < 1$ if and only if*

$$[V_2(x,y) - V_2(y,x)]\,(y - x) \leq 0 \qquad \forall (x,y),(y,x) \in X \times X. \tag{11}$$

When V is twice differentiable the (11) implies $V_{12}(x,x) \geq V_{22}(x,x)$.

Example 2. Consider $V(x,y) = ax + by - (1/2)\,Ax^2 - y^2 - xy - (1/2)\,Bxy^2$ defined on any set $D \subseteq [0,1] \times [0,1]$. For the following set of parameters V satisfies Assumption A3: $A \geq 1/2$, $B \geq -2$, and $(2A-1) \geq B(B+2)$. This function also satisfies Condition (11), and is therefore acyclic because of Proposition 10. Furthermore, when $B > 0$, V(x,y) is submodular, whereas for $B < 0$ V is neither sub- nor supermodular. This proves the claim that acyclicity does not imply supermodularity. ◇

4.4 Low–period cycles

As an application of our approach, which we regard as a field in which work has yet to be done, in this section we tackle the problem of excluding not all cycles, but just those of high period. Again, we have to restrict ourselves to the case of X a subset of the real line. A first step in this direction was Theorem 3.2 on the ranking of the cycles of a relation. But the theorem has the practical problem that it is difficult, for many relations, to exclude cycles of higher order once one has ascertained the presence of period–two cycles. Moreover, the discount factor plays no role, in the sense that the acyclicity of a relation is independent of the magnitude of δ.

One could then think to study another relation, that we could call P_θ^2. This relation would be satisfied by the second iterate of τ. If P_θ^2 was acyclic, then we would know that τ has no periodic points of period 4. But this, by Sarkovsky's theorem, would exclude periodic points of any period other than 2. Moreover, it is our hope that further work in this direction may highlight the role of the discount factor δ.

Let us define a new function V^2, defined as

$$V^2(x,y) = \max_u V(x,u) + \delta V(u,y) \quad \text{such that } (x,u),(u,y) \in D. \tag{12}$$

Now, let $D^2 = \{(x,y)|$ there is a u such that $(x,u),(u,y) \in D\}$. A little reflection shows that, if D satisfies A2, then D^2 satisfies A2 too. On the other hand, if V satisfies A3, then V^2 satisfies A3. Then the problem

$$W(x) = \max_{y \in D^2(x)} V^2(x,y) + \delta^2 W(y) \tag{13}$$

yelds a well-behaved function, which is the second iterate of τ.

The relation P_θ^2 is defined as

$$yP_\theta^2 x \Leftrightarrow V^2(x,(1-\theta)x+\theta y) + \delta W((1-\theta)x+\theta y) \leq V^2(x,y) + \delta W(y),$$

where $W(\cdot)$ is some concave function. To exclude periodic points of period 4, we need to check the acyclicity of V^2. Moreover, since V^2 depends from δ, we hope to exploit this dependence.

Example 3. Set

$$V(x,y) = -\frac{y^3}{3} - byx + k\frac{x^2}{2} - \frac{y^2}{2},$$

with $X = [0,1]$ and $D = X \times X$.

One may check that V is concave iff $k \leq -b^2$. If $b > 1$, then using Proposition 10 we find that V is not additively θ-acyclic for any θ in (0,1).

Calculating V^2 yelds

$$V^2(x,y) = k\frac{x^2}{2} - \delta\frac{y^3}{3} - \delta\frac{y^2}{2},$$

and again using Proposition 10 one finds that V^2 is acyclic. Hence, although V is cyclic, the only possible cycles are of period two. This result is, again, valid for any value of the discount factor.

Theorem 3.2, applied to this case, produces cumbersome calculations that we have not tried to manage. ◇

Example 4. Set

$$V(x,y) = -x^2 - y - xy - y^3x^2(yx+1)$$

with $X = [0,1]$ and $D = X \times X$.

One readily calculates that V is cyclic, but computing V^2, one finds that $V_2(x,y) = -x^2 - \delta y$ is acyclic. ◇

The preceding examples all had the feature that the value of u maximizing expression (12) was on the boundary of the admissible values. In fact, a little reflection shows that, if such a feature is verified, then V^2 is acyclic, since it is the sum of two functions respectively of x and y. But it is not true that, in case the value of u is internal to the feasible set, then V^2 is cyclic. This is the argument of the following

Example 5. Set $V(x,y) = \mu x^2 - y - xy + \mu x$, with $X = [0,1]$, $D = X \times X$ and μ a positive parameter. The same calculations performed in the example above yeld that V is cyclic. Working out V^2, one finds that if $\mu > 1 + (2/\delta)$ then u is interior to X, and

$$u = 1 - \frac{\delta y + x + 1}{\delta \mu}.$$

Substituting, after long-lasting algebra one gets

$$V_2^2(x,y) = -\delta,$$

which shows that V^2 is acyclic. ◇

Note that the function V used in this last example is strictly concave in the first variable rather than in the second. In fact, this hypothesis could have been made in A3 with no harm for the properties of the policy function, except for the case in which the discount factor equals 0.

References

[1] J.P. Aubin and A. Cellina, "Differential Inclusions", *Springer-Verlag* 1984.

[2] M. Boldrin and L. Montrucchio, "On the Indeterminacy of Capital Accumulation Paths", *Journal of Economic Theory* 40 (1986), 26–39.

[3] M. Boldrin and L. Montrucchio, "Acyclicity and Stability for Intertemporal Optimization Models", *International Economic Review* 29(1) (1988), 137–146.

[4] M. Boldrin and L. Montrucchio, "Acyclicity and Dynamical Stability: Generalizations and Economic Applications", *MIMEO* 1990.

[5] J. Hale, "Asymptotic Behavior of Dissipative Systems", *American Mathematical Society* Rhode Island, Providence 1988.

[6] J. P. LaSalle, "The Stability of Dynamical Systems", *Society for Industrial and Applied Mathematics* Philadelphia, Pennsylvania 1976.

[7] L. Montrucchio, "Optimal Decisions Over Time and Strange Attractors: an Analysis by the Bellman Principle", *Mathematical Modelling* 7 (1986), 341–352.

[8] A. N. Sarkovsky, "Coexistence of Cycles of a Continuous Map of the Line into Itself", *Ukr. Mat. Z.* 16 (1964), 61–71.

[9] S. Smale, "Differential Dynamical Systems", *Bulletin of the American Mathematical Society* 1967

[10] N. Stokey and R. Lucas and R. Prescott, "Recursive Methods in Economic Dynamics", *Harvard University Press* 1989.