Games of Redistributive Politics Are Equivalent to All-Pay Auctions With Consolation Prizes

Nicola Persico*

University of Pennsylvania

Abstract

Myerson (1993) introduces a model of redistributive politics where candidates offer different transfers to different voters, in the attempt to win an election. He compares, under different electoral systems, the incentive for candidates to create inequalities among otherwise homogeneous voters.

Different electoral systems can fruitfully be reinterpreted as different *all-pay auctions with consolation prizes.* These are auctions where each bidder enters one bid and receives (at most) one prize, with the j-th highest bidder receiving the j-th most valuable prize. We show that, given any Rank-Scoring rule, the equilibrium in an election under that rule is equivalent to the equilibrium in an all-pay auction with appropriately chosen prizes.

1 Introduction

In games of redistributive politics, candidates compete for election by making binding promises to each voter, concerning the allocation of a budget if elected. Myerson

*Department of Economics, 3718 Locust Walk, Philadelphia, PA 19104-6297. Email: persico@ssc.upenn.edu. Web: http://www.ssc.upenn.edu/~persico. This paper was motivated by a discussion with Paul Klemperer, and is a spinoff of ongoing research joint with Alessandro Lizzeri.

(1993) considers games of redistributive politics under a wide variety of electoral rules, and compares these rules in terms of the redistributional inequality that prevails at equilibrium. Lizzeri (1999) and Lizzeri and Persico (1998, 2000) use Myerson's model to study budget deficits and to discuss public goods provision, respectively. In this short note we show that the equilibrium of games of redistributive politics can be interpreted as the equilibrium of auctions with complete information. Thus, the intuition that comes from the auction setting can be fruitfully transferred to electoral games.

We consider *all-pay auctions with consolation prizes.* An all-pay auction with consolation prizes is a generalization of the all-pay auction, where each bidder puts in one bid and receives (at most) one prize, with the j-th highest bidder receiving the j-th most valuable prize. We show that a game of redistributive politics under a variety of electoral rules (any Rank-Scoring rule) is equivalent to an all-pay auction with appropriately chosen prizes, in the sense that their (symmetric) equilibrium is the same.

This might be surprising given the difference in their strategy spaces. In games of redistributive politics, candidates' strategies are redistributive platforms — allocations of a budget among subset of voters. In contrast, a strategy in an all-pay auction is one bid. However, the symmetric equilibrium in an all-pay auctions is a mixed strategy. The c.d.f. of this mixed strategy translates, in the game of redistributive politics, into a function describing how many people receive less than x transfers by a candidate in equilibrium. This notion can be translated into the (perhaps more familiar) notion of a Lorenz curve of a redistributive policy.

For example, consider an election between 3 candidates under a popular Rank-Scoring rule, Borda rule. Under this rule, each voter must indicate a ranking of the 3 candidates on a ballot; then this voter's ballot gives the top-ranked candidate 1 point, the second-ranked candidate 1/2 points, and the third candidate 0 points. Each candidate's score is the average of all points received from all voters. The winner of the election is the candidate with the highest score. We show that the equilibrium in this election is equivalent to the equilibrium in an all-pay auction with 3 bidders, where the highest bidder receives a prize of 2, the second highest receives a prize of 1, and the lowest bidder receives zero.

2 Model (Myerson (1993))

2.1 Economy and Agents

There are N candidates. There is a continuum of measure 1 of voters. Voters have no a priori preference for any candidate, and have linear utility over money.[1]

If elected, a candidate will dispose of a budget of 1 (corresponding to one unit of money per voter). Before the election, candidates make binding promises to each voter, concerning the allocation of the budget if elected. A candidate can offer different transfers to different voters. A pure strategy for a candidate specifies a promise of a transfer to each voter. Formally, a pure strategy is a function $\Phi : V \to [0, +\infty)$ that satisfies the balanced budget condition $\int_V \Phi(v)dv = 1$.

Definition 1 *A* ***Rank-Scoring rule*** *with N candidates is an electoral system characterized by an ordered sequence of N numbers, $1 = s_1 \geq s_2 \geq ... \geq s_N = 0$. In the election, each voter must indicate a ranking of the N candidates on a ballot; then this voter's ballot gives the top-ranked candidate s_1 points, the second-ranked candidate s_2 points, and so on, with the j-th candidate getting s_j points. Each candidate's score is the average of all points received from all voters. The winner of the election is the candidate with the highest score.*

Because a candidate's promise is only relevant if he gets elected, each voter's optimal behavior is to rank candidates according to who promises her more transfers. There are two stages of the game:

Stage 1 Candidates choose offers to voters simultaneously and independently.

[1] This assumption is made solely to simplify the notation and presentation of the results; all the results hold for any increasing utility function.

Stage 2 Each voter v gets promises $\Phi_1(v), \Phi_2(v), ..., \Phi_N(v)$ from candidates. After observing the offers, voter v ranks candidates on the ballot according to who promises her more. If the voter gets the same highest offer from two or more candidates, she randomizes with equal probability.

2.2 No Equilibrium in Pure Strategies

Consider the case with two candidates ($N = 2$). Suppose at equilibrium candidate 2 employs a pure strategy $\Phi_2(v)$. Then, candidate 1 could respond by offering $\Phi_2(v) + \varepsilon$ to 99% of the v's, and finance this policy by offering zero to 1% of the voters. This strategy delivers a vote share of 99% to candidate 1, which cannot be an equilibrium. Thus, the equilibrium must be in mixed strategies. The same logic applies to all rank-scoring rules with N candidates.

2.3 Mixed Strategies as Lorenz Curves

We discuss the case where the offers of transfers made by candidate i to voters are realizations of the same random variable with c.d.f. F_i. Of course, the fact that offers are realizations of the same random variable does not mean that each voter gets the same offer. Also note that before voting, each voter observes his realized promises, not random variables.

Because there are infinitely many voters, F_i will be the empirical distributions of offers in the electorate. $F_i(x)$ is the fraction of voters who, if i is elected, receive transfers below x. By manipulating F_i, candidate i is able to target transfers to sections of the populations. The budget constraint is $\int_0^\infty x dF_i(x) = 1$.

It is appropriate to think of $F_i(x)$ as defining a Lorenz curve, a descriptive statistics which describes the distribution of wealth in the population.

3 Equivalence

Definition 2 *An All-Pay auction with N bidders and consolation prizes, is an auction where all bidders pay their bid; the highest bidder receives π_1, the second highest bidder receives π_2, and so on, with the j-th highest bidder receiving π_j and the lowest bidder receiving zero.*

Elections under general Rank-Scoring rules are equivalent to all-pay auctions with suitably chosen prizes.

Proposition 1 The equilibrium in any N-candidate election game with a Rank-Scoring rule $(s_1, s_2, ..., s_N)$ is equivalent to the equilibrium in an all-pay auction with N bidders and consolation prizes $(\pi_1, \pi_2, ..., \pi_N)$, where $\pi_j = N s_j / \sum_{k=1}^{N} s_k$.

Proof: Let F^* denote the equilibrium of the election. Suppose candidate 1 offers x to a voter. Then, the candidate receives s_1 points with probability $\left[F^*(x)\right]^{N-1}$, s_2 points with probability $(N-1)\left[1 - F^*(x)\right]\left[F^*(x)\right]^{N-2}$, and in general the candidate receives s_j points with probability

$$p_j\left(F^*(x)\right) = \binom{N-1}{j-1}\left[1 - F^*(x)\right]^{j-1}\left[F^*(x)\right]^{N-j}$$

At equilibrium, candidate 1 maximizes his expected score subject to the budget constraint

$$\max_{F_1} \int_0^\infty \sum_{j=1}^{N} s_j p_j\left(F^*(x)\right) dF_1(x) \text{ s.t. } \int_0^\infty x dF_1(x) = 1.$$

This problem can be rewritten as

$$\begin{aligned} & \max_{F_1} \int_0^\infty \sum_{j=1}^{N} s_j p_j\left(F^*(x)\right) dF_1(x) + \lambda^* \left[1 - \int_0^\infty x dF_i(x)\right] \\ = \;& \lambda^* \max_{F_1} \int_0^\infty \left(\sum_{j=1}^{N} \frac{s_j}{\lambda^*} p_j\left(F^*(x)\right) - x\right) dF_1(x) + \lambda^*. \end{aligned} \tag{1}$$

Up to a linear affine transformation involving λ^*, the above expression can be interpreted as the payoff of bidder 1, who bids according to a mixed strategy F_1, in an all-pay auction with prizes $\pi_j = s_j/\lambda^*$. Thus, the symmetric equilibria of the two games are equivalent.

To determine the value of λ^*, we use our knowledge of the all-pay auction. We know that, given any value of λ^*, bidders dissipate their surplus entirely in the all-pay auction. So, at equilibrium the term involving the maximization equals zero, and the value of the Lagrangean for the election game equals λ^*. Therefore, λ^* must equal the expected score of candidates in the election. By symmetry, the expected score at equilibrium is $\frac{1}{N}\sum_{k=1}^{N} s_k = \lambda^*$. Therefore, the j-th prize in the auction has value $\pi_j = \frac{s_j}{\lambda^*} = \frac{Ns_i}{\sum_{k=1}^{N} s_k}$ ∎

To characterize the symmetric equilibrium in the all-pay auction, notice that each bidder must be willing to mix at equilibrium. Thus, the payoff for any bid x in the support of the mixed strategy must equal zero, the expected payoff from the game. Using expression (1),

$$0 = \sum_{j=1}^{N} \frac{s_j}{\lambda^*} p_j\left(F^*\left(x\right)\right) - x = \sum_{j=1}^{N} \frac{Ns_j}{\sum_{k=1}^{N} s_k} p_j\left(F^*\left(x\right)\right) - x \tag{2}$$

for all x in the support of the mixed strategy. Since the expressions for λ^* and p_j are known, this equality characterizes the equilibrium strategy F^*.[2]

The total value of the prizes awarded in the auction sums up to N, which equals the sum of the budget constraints of candidates in the electoral game. The reason for this is that candidates in the election use up all their budget of 1, so at the equilibrium in the all-pay auction it must be that the each bidders' expected payment also equals 1. In total, bidders are expected to spend N dollars, and since the bidders' expected profit at equilibrium is zero, it must be that the total expected revenue to be divided among the bidders equals N.

Example [1] **Elections With Two Candidates**. With two candidates, the definition of rank-scoring rule implies $s_1 = 1, s_2 = 0$. Therefore, this election is equivalent to an all-pay auction with first prize equal to 2, and no consolation prize. The symmetric

[2] This proves Theorem 2 of Myerson (1993).

equilibrium strategy F^* in such an auction is a uniform between zero and 2. This proves Theorem 1 in Myerson (1993).

Example [2] **Election With N Candidates and Plurality Rule**. Plurality rule is a rule where the voter casts only one vote, and the candidate receiving more votes is elected. This is a rank-scoring rule where $s_1 = 1$, and $s_2 = s_3 = ... = s_N = 0$. Therefore, this election is equivalent to an all-pay auction with first prize equal to N, and no consolation prizes. To find the equilibrium strategy F^*, rewrite equation (2) as

$$0 = \frac{Ns_1}{s_1} p_1 \left(F^*(x)\right) - x = N \left[F^*(x)\right]^{N-1} - x,$$

whence $F^*(x) = (x/N)^{1/(N-1)}$ for $x \in [0, N]$.

Example [3] **Borda Rule With 3 Candidates.** In this election, each voter must indicate a ranking of the three candidates on a ballot; then this voter's ballot gives the top-ranked candidate 1 point, the second-ranked candidate 1/2 points, and the third candidate getting 0 points. Each candidate's score is the average of all points received from all voters. The winner of the election is the candidate with the highest score. This is a rank-scoring rule where $s_1 = 1$, $s_2 = 1/2$, $s_3 = 0$. Therefore, this election is equivalent to an all-pay auction with first prize equal to $3s_1/(s_1 + s_2) = 2$, second prize equal to $3s_2/(s_1 + s_2) = 1$, and third prize equal to zero.

Equation (2) specializes to

$$0 = 2 \cdot [F^*(x)]^2 + 1 \cdot 2F^*(x) [1 - F^*(x)] - x.$$

Solving for F^* we get $F^*(x) = x/2$ for $x \in [0, 2]$.

3.1 Inequality

A measure of the inequality of electoral redistribution is the *maximal offer* than any voter may receive at equilibrium. With plurality voting, the maximal offer equals the number of candidates N; that is, there are some voters who receive zero transfers, and other voters who receive transfers of size N. In contrast, with Borda rule and three candidates the maximal offer equals 2. This suggests that Borda rule limits the candidates' incentives towards tactical redistribution, and generates less inequality, relative to plurality voting. It is interesting to understand what determines the maximal offer in an electoral system.

This is clear once we interpret the maximal offer as the maximum of the support of the equilibrium strategy in the all-pay auction. In an all-pay auction with consolation prizes, when bidders bid at the maximum they receive π_1 for sure. At equilibrium, bidders make zero expected profits, so the maximum of the support must equal π_1. Since $\pi_1 = Ns_1/\sum_{k=1}^{N} s_k = N/\sum_{k=1}^{N} s_k$, then this is also the maximal offer; the closer $s_2, ..., s_{N-1}$ are to 1, the lower the maximal offer. The highest possible value of the maximal offer is thus N, which is achieved under plurality rule. The smallest possible value is $N/(N-1)$.

Intuitively, when $s_2, ..., s_{N-1}$ are close to zero then also $\pi_2, ..., \pi_{N-1}$ are close to zero. In this case, only π_1 is large, and has size almost N. Then, bidders have a large incentive to overbid their opponents, and the maximum of the support of the equilibrium strategy is large. In contrast, when $s_2, ..., s_{N-1}$ are close to 1 then also $\pi_2, ..., \pi_{N-1}$ are close to 1. When a player in an auction wins almost the same prizes unless he is the lowest bidder, he will not have a large incentive to overbid his opponents, and the maximum of the support of the equilibrium strategy will be small.

4 Conclusion

We have shown that the equilibrium in a game of redistributive politics where candidates compete under any rank-scoring rule is equivalent to the equilibrium in an all-pay auction with appropriately chosen prizes. The prizes in the auction are chosen to sum up to the number of candidates. The inequality in the auction prizes (the difference between first and second, second and third prize, and so on) determines the maximal offer that bidders make in the auction, and therefore the degree of inequality induced by the corresponding rank-scoring rule in the election.

References

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[3] Lizzeri, Alessandro and Persico, Nicola. "The Drawbacks of Electoral Competition." CARESS Working Paper # 00-02, University of Pennsylvania, 2000.

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5 Reasoned Bibliography

The earliest attempts at modeling electoral games with redistribution gave rise to Condorcet cycles (much as is shown in Section 2.2: given any redistribution of income in pure strategies, there is always another redistribution of income in pure strategies that beats it). Thereby, people more or less threw up their hands in dispair for a long time. Myerson's model solves this problem with mixed strategies.

In contrast, Lindbeck and Weibull (1987) resort to "**probabilistic voting**," where the probability that a voter votes for a candidate is smoothly increasing in the amount that candidate offers. This smoothness allows, under some conditions, to recover a pure strategy equilibrium. In my opinion, that model is not very tractable; for instance, I know of no extension to more than two candidates. Nonetheless, the model is popular (see Dixit and Londregan (1996,1998)).

Building on the model of Myerson, there are three papers. Lizzeri (1999), Lizzeri and Persico (1998) and Lizzeri and Persico (2000).

In Lizzeri and Persico (1998), we take Myerson's model and add a public good. The public good costs all the budget to provide, and is efficient, in the sense that it gives every voter a utility larger than 1. However, the public good cannot be targeted to subsets of the voters, in the way that transfers can. This gives rise to a tradeoff between efficiency and targetability, and a consequent scope for inefficiency. We study how different political systems deal with this inefficiency.

In Lizzeri and Persico (2000) we provide a model that challenges the conventional view that in politics, just as in economic markets, competition between politicians is a force that pushes towards efficiency. In the model, candidates can promise to provide a public good or to engage in redistributive politics. We show that the more intense is competition (measured by an increase in the number of candidates) the greater the inefficiency. This is because the tendency to focus on policies that provide particularistic benefits increases with the number of candidates at the expense of policies that benefit the population at large.

So, that was my sales pitch. I have something more cooking where I use this model, but I'll let you know when it's done.

Finally, a historical curiosity. The model with two candidates (Example 1) existed in the literature as a game called "Colonel Blotto." I quote Myerson (1993):

> Game theorists since Gross and Wagner (1950) have found such uniform distributions in equilibria of "Colonel Blotto" games, where each of the two competitors divides a fixed supply of resources over a set of battlefields. [...] To apply the "Colonel Blotto" terminology to our context, we can reinterpret the battlefields as voters, for whose votes the candidates compete with their offers.

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